Recitation 8. May 4

Focus: positive definite/semidefinite matrices, singular value decomposition, pseudo-inverse

If all the eigenvalues (or equivalently, the pivots) of a symmetric matrix S are positive/non-negative, then S is called:

$$\boxed{\textbf{positive definite/semidefinite}}$$

We have:

$$S \text{ positive definite} \quad \Leftrightarrow \quad \boldsymbol{v}^T S \boldsymbol{v} > 0$$
$$S \text{ positive semidefinite} \quad \Leftrightarrow \quad \boldsymbol{v}^T S \boldsymbol{v} \geq 0$$

for any vector $\boldsymbol{v} \neq 0$. The quantity $\boldsymbol{v}^T S \boldsymbol{v}$ is called the $\boxed{\textbf{energy}}$ of $\boldsymbol{v}$.

The $\boxed{\textbf{Singular Value Decomposition}}$ (SVD) of a matrix A is a way of writing it as:

$$\boxed{A = U \Sigma V^T}$$

where U and V are orthogonal matrices, and Σ is diagonal. If we let:

$$U = \left[\ \boldsymbol{u}_1 \mid \dots \mid \boldsymbol{u}_m\ \right] \quad \text{and} \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & \dots \\ 0 & \ddots & 0 & 0 & \dots \\ 0 & 0 & \sigma_r & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad V = \left[\ \boldsymbol{v}_1 \mid \dots \mid \boldsymbol{v}_n\ \right]$$

then the SVD is a way of writing A as a sum of rank 1 matrices:

$$\boxed{A = \sum_{i=1}^{r} \boldsymbol{u}_i \sigma_i \boldsymbol{v}_i^T}$$

You may compute the SVD by letting:

- the **right singular vectors** $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ be the eigenvectors of $A^T A$
- the **left singular vectors** $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m$ be the eigenvectors of AA^T
- the **singular values** $\sigma_1, \dots, \sigma_r$ be the square roots of the non-zero eigenvalues of $A^T A$ or AA^T

The $\boxed{\textbf{pseudo-inverse}}$ of A is defined by:

$$\boxed{A^+ = V \Sigma^+ U^T}$$

where Σ^+ has diagonal entries $\frac{1}{\sigma_i}$ instead of σ_i. It is useful because:

$$\boxed{\text{the closest } A\boldsymbol{v} \text{ can be to a vector } \boldsymbol{b} \text{ is achieved for } \boldsymbol{v} = A^+ \boldsymbol{b}}$$

Moreover:

$$AA^+ \text{ is the projection matrix onto } C(A)$$
$$A^+ A \text{ is the projection matrix onto } C(A^T)$$

Also, the SVD allows us to define the $\boxed{\textbf{polar decomposition}}$ of $A = U \Sigma V^T$ as:

$$\boxed{A = QS}$$

where $Q = UV^T$ is orthogonal, and $S = V \Sigma V^T$ is symmetric.

1. Consider the matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

- Say which of them is positive definite, positive semidefinite, or neither.
- Write down the energy of an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with respect to either of these matrices. Setting the energy equal to 1 gives rise to a conic. What kind of conic is it, in each of the three cases?

Solution: The characteristic polynomial of A is $(1-\lambda)(3-\lambda) - 1 = \lambda^2 - 4\lambda + 2$. Thus the eigenvalues are $2 \pm \sqrt{2}$, so A is positive definite. The energy is:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 2xy + 3y^2 = (x+y)^2 + 2y^2$$

Setting $(x+y)^2 + 2y^2 = 1$ is the equation of an ellipse.

The characteristic polynomial of B is $(3-\lambda)(-2-\lambda)$. Thus the eigenvalues are 3 and -2, so B is not positive (semi)definite. The energy is:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3x^2 - 2y^2 =$$

Setting $3x^2 - 2y^2 = 1$ is the equation of a hyperbola.

The characteristic polynomial of C is $(1-\lambda)^2 - 1$. Thus the eigenvalues are 2 and 0, so C is positive semidefinite. The energy is:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 2xy + y^2 = (x+y)^2$$

Setting $(x+y)^2 = 1$ is the equation of a pair of lines.

2. Consider the matrix

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

- Compute the Singular Value Decomposition of A.
- Compute the pseudo-inverse A^+. Then compute the inverse A^{-1} by another method. How do they compare?

Solution: First we calculate $A^T A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$. Next we diagonalize this matrix:

$$\det(A^T A - \lambda I) = \lambda^2 - 10\lambda + 16 \qquad \Rightarrow \qquad \lambda = 8 \text{ or } 2$$

Moreover, being symmetric, $A^T A$ has orthonormal eigenvectors:

$$N \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} = \mathbb{R} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad N \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \mathbb{R} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

for $\lambda = 8$ and $\lambda = 2$, respectively. We therefore set:

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

Then to find the $\boldsymbol{u}_i$, we use the formula:

$$\boldsymbol{u}_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \frac{1}{2\sqrt{2}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{u}_2 = \frac{Av_2}{\sigma_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So $U = I$ and the full SVD is:

$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

The pseudo-inverse is

$$A^+ = V\Sigma^+U^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$$

Equivalently, we can use the formula for the inverse of a 2×2 matrix to see that:

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$$

The phenomenon here is that when a matrix is invertible, its inverse coincides with its pseudo-inverse.

3. - Express the function:

$$\frac{3x^2 + 2xy + 3y^2}{x^2 + y^2} \tag{1}$$

in the form $\frac{\boldsymbol{v}^T S \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}}$, where $\boldsymbol{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and S is a certain symmetric matrix that you are free to choose. Compute the maximum of (1) in terms of the eigenvalues of S. For what values of (x, y) is the maximum achieved?

- Find the maximum of the function:

$$\sqrt{\frac{(x + 4y)^2}{x^2 + y^2}}$$

by expressing it in the form $\frac{\|A\boldsymbol{v}\|}{\|\boldsymbol{v}\|}$ for a suitable matrix A, and then invoking the singular values of A.

Solution:
We can express the numerator as:

$$3x^2 + 2xy + 3y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

so the required expression is $\frac{\boldsymbol{v}^T S \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}}$ for $S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. The maximum of the expression is given by the largest eigenvalue of the matrix S, and the maximum is achieved at the corresponding eigenvector. The matrix S has eigenvalues $\lambda = 4$ and $\lambda = 2$ so the maximum is 4. The corresponding eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So the minimum is achieved whenever $x = y$, i.e. $\boldsymbol{v}$ is a multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We can express $(x + 4y)^2 = \|A\boldsymbol{v}\|^2$ for the 1×2 matrix $A = \begin{bmatrix} 1 & 4 \end{bmatrix}$. The expression is maximized by the largest singular value of A (in absolute value), which is the square root of the largest eigenvalue of:

$$A^T A = \begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix}$$

It's easy to see that the eigenvalues of $A^T A$ are 17 and 0, so the maximum is $\sqrt{17}$.